Classical Moduli Space (Flat directions)

$F < N$ $\quad 0 = D^a = g\left(\phi^{*in} T^{am}{}_n \phi_{mi} - \bar{\phi}^{in} T^{am}{}_n \bar{\phi}^*_{mi}\right)$

$d^n_m \equiv \phi^{*in} \phi_{mi}; \quad \bar{d}^n_m = \bar{\phi}^{in} \bar{\phi}^*_{mi}$

$N \times N$ positive sem-definite matrix of rank F

$$D^a = T^{am}{}_n \left(d^n_m - \bar{d}^n_m\right) = 0$$

d^n_m can be diagonalized by an $SU(N)$ gauge transformations

$$U^\dagger d^n_m U$$

$N - F$ zero eigenvalues

F positive semi definite

in the same basis $\bar{d}$ is also diagonal

to satisfy $D^a = 0$

$$\begin{pmatrix} v_1^2 & & & & & \\ & v_2^2 & & & & \\ & & \ddots & & & \\ & & & v_F & & \\ & & & & 0 & \\ & & & & & 0 \\ & & & & & & 0 \end{pmatrix} - \begin{pmatrix} \bar{v}_1^2 & & & & & \\ & \ddots & & & & \\ & & \bar{v}_? & & & \\ & & & 0 & & \\ & & & & \bar{v}_p^2 & \\ & & & & & \ddots \\ & & & & & & \bar{v}^2 \\ & & & & & & & 0 \end{pmatrix} = \alpha I$$

$v_i^2 \geq 0$

$\bar{v}_i^2 \geq 0$ $\quad \Rightarrow \alpha = 0 \quad \Rightarrow \bar{v}_i^2 = v_i^2$

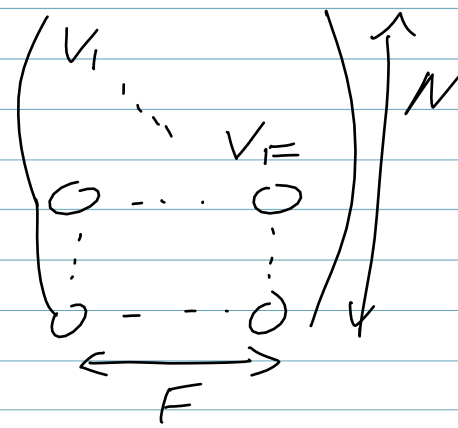
Flavor Transformations

$\phi_{mi} \to \phi_{mi} V^i_{\ j}$

$d^n_m \to V^{*j}_{\ \ k} \phi^{*kn} \phi_{mi} V^i_{\ j}$

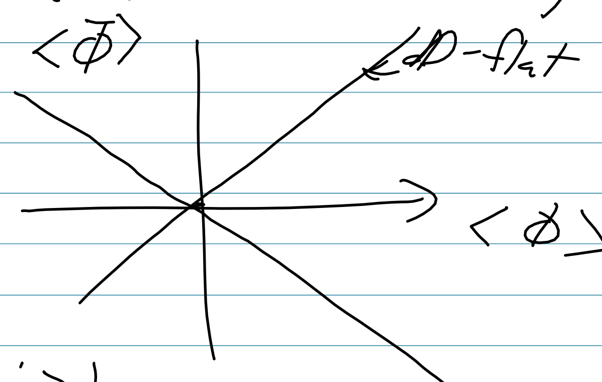
$= \phi^{*in} \phi_{mi} = d^n_m$

$\langle \bar\phi^* \rangle = \langle \phi \rangle =$



$F < N$ generically $SU(N) \to SU(N-F)$

moduli space



ex $\quad V_1 = V \qquad V_i = 0 \quad i > 1$

$SU(N) \to SU(N-1)$

$SU(F) \times SU(F) \Rightarrow SU(F-1) \times SU(F-1)$

broken gauge generators:

$N^2 - 1 - ((N-1)^2 - 1) = N^2 - 1 - (N^2 - 2N + 1 - 1)$

$= 2N - 1 = 2(N-1) + 1$

$$F \geq N$$

$$d - \bar{d} = \rho I$$

$$d = \begin{pmatrix} V_1^2 & & \\ & \ddots & \\ & & V_N^2 \end{pmatrix} \qquad \bar{d} = \begin{pmatrix} \bar{V}_1^2 & & \\ & \ddots & \\ & & \bar{V}_N^2 \end{pmatrix}$$

$$V_i^2 = \bar{V}_i^2 + \rho$$

$$\phi = \underbrace{\begin{pmatrix} V_1 & & & & 0 & \cdots & 0 \\ & V_2 & & & 0 & & \vdots \\ & & \ddots & & \vdots & & \vdots \\ & & & V_N & 0 & & 0 \end{pmatrix}}_{F} \Big\updownarrow N$$

$$\bar{\phi} = \underbrace{\begin{pmatrix} \bar{V}_1 & & \\ & \ddots & \\ & & \bar{V}_N \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}_{N} \Big\updownarrow F$$

generator basis

$$G^A = \{ X^0, X_1^\alpha, X_2^\alpha, T^a \}$$

$$a = 1, \ldots (N-1)^2 - 1 \qquad A = 1, \ldots N^2 - 1$$

$$\alpha = 1, \ldots, N-1$$

$$X^0 = \frac{1}{\sqrt{2}\sqrt{N^2 - N}} \begin{pmatrix} N-1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$$

$$X_1^\alpha = \frac{1}{2} \begin{pmatrix} 0 & \cdots & 0 & \overset{\alpha+1}{1} & 0 & \cdots & 0 \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \\ 1 & & & \text{\Large 0} & & & \\ 0 & & & & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{pmatrix} \quad (\text{row } \alpha+1)$$

$$X_2^\alpha = \frac{1}{2} \begin{pmatrix} 0 & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ -i & & & & \text{\Large 0} & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{pmatrix}$$

$$X^+ = \frac{1}{\sqrt{2}}(X_1 - i X_2) \qquad X^- = \frac{1}{\sqrt{2}}(X_1 + i X_2)$$

$$G^A G^A = X^0 X^0 + X^{+\alpha} X^{-\alpha} + X^{-\alpha} X^{+\alpha} + \ldots T^a$$

$$X^{+\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \cdots 0 \; 1 \; 0 \; 0 \cdots 0 \\ \\ 0 \\ \\ \end{pmatrix}$$

$$X^{-\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{matrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{matrix} & 0 \end{pmatrix}$$

$$\phi \rightarrow \langle \phi \rangle + \phi$$

$$\sum_A G^A \langle \phi \rangle = X^0 \langle \phi \rangle + \sum_\alpha X^{-\alpha} \langle \phi \rangle$$

$$\sum_A \langle \phi^* \rangle G^A = \langle \phi^* \rangle X^0 + \sum_\alpha \langle \phi^* \rangle X^{+\alpha}$$

$$G^A \lambda^A = X^0 \Lambda^0 + X^{+\alpha} \Lambda^{+\alpha} + X^{-\alpha} \Lambda^{-\alpha} + T^a \Lambda^a$$

$$Q = \left(\begin{array}{c|c} \omega^0 & \psi^{\Sigma} \\ \hline \omega^\alpha & Q' \end{array} \right) \qquad \bar{Q} = \left(\begin{array}{c|c} \bar{\omega}_0 & \bar{\omega} \\ \hline \bar{\Psi} & \bar{Q}' \end{array} \right)$$

(Q: lower block $N-1$ rows, $F-1$ columns; $\bar{Q}$: lower block $F-1$ rows, $N-1$ columns)

$$\phi^* G^A Q \lambda^A - \lambda^{\dagger A} Q^\dagger G^A \phi$$

$$\mathcal{L}_{\text{yuk mass}} = -\sqrt{2} g \begin{pmatrix} \left(\langle\phi^*\rangle X^0 \Lambda^0 + \langle\phi^*\rangle X^{+\alpha} \Lambda^{+\alpha}\right) Q \\ -\bar{Q}\left(\Lambda^0 X^0 \langle\bar{\phi}^*\rangle + \Lambda^{-\alpha} X^{-\alpha} \langle\bar{\phi}^*\rangle\right) \\ + h.c. \end{pmatrix}$$

$$= -gv \begin{pmatrix} \frac{N-1}{\sqrt{N^2-N}}\left(\omega^0 \Lambda^0 - \bar{\omega}^0 \Lambda^0\right) \\ + \Lambda^{+\alpha} \omega^\alpha - \Lambda^{-\alpha} \bar{\omega}^\alpha \\ + h.c. \end{pmatrix}$$

$$\begin{matrix} & \Lambda^0 & \omega^0 & \bar{\omega}^0 \end{matrix}$$

$$\frac{N-1}{\sqrt{N^2-N}} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{matrix} \Lambda^0 \\ \omega^0 \\ \bar{\omega}^0 \end{matrix}$$

$$\det = -\lambda \cdot \lambda^2 - 1 \begin{vmatrix} 1 & 0 \\ -1 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -\lambda \\ -1 & 0 \end{vmatrix}$$
$$= -\lambda^3 + \lambda + \lambda$$
$$= -\lambda(\lambda^2 - 2)$$

Q', $\bar{Q}'$, ψ, $\bar{\psi}$, $\frac{\omega^0 + \bar{\omega}^0}{\sqrt{2}}$ $\quad m = 0$

Dirac
$$\left(\Lambda^0, \frac{\omega^0 - \bar{\omega}^0}{\sqrt{2}}\right) \quad m = \sqrt{2}\sqrt{\frac{N-1}{N}}\, gv$$

$$(\Lambda^{+\alpha}, \omega^\alpha) \quad m = gv$$
$$(\Lambda^{-\alpha}, \bar{\omega}^\alpha)$$

$$\phi = \left(\begin{array}{c|c} h & \sigma \\ \hline H^\alpha & \phi' \end{array}\right) \updownarrow N-1, \quad \overset{\longleftrightarrow}{F-1} \qquad \bar{\phi} = \left(\begin{array}{c|c} \bar{h} & \bar{H}^\alpha \\ \hline \sigma & \bar{\phi}' \end{array}\right) \updownarrow F-1, \quad \overset{\longleftrightarrow}{N-1}$$

$$V = \frac{1}{2} D^A D^A$$

$$\frac{1}{g} D^A = \underbrace{\langle\phi^*\rangle G^A \langle\phi\rangle - \langle\bar\phi\rangle G^A \langle\bar\phi^*\rangle}_{=0 \text{ since D-flat}}$$
$$+ \langle\phi^*\rangle G^A \phi + \phi^* G^A \langle\phi\rangle - \langle\bar\phi\rangle G^A \bar\phi^* + \bar\phi G^A \langle\bar\phi^*\rangle$$
$$+ \phi^* G^A \phi - \bar\phi G^A \bar\phi^*$$

$$V_{mass} = \frac{g^2}{2}\Big(\langle\phi^*\rangle (X^0 + X^{+\alpha})\phi + \phi^*(X^0 + X^{-\alpha})\langle\phi\rangle$$
$$- \langle\bar\phi\rangle (X^0 + X^{+})\bar\phi^* - \bar\phi (X^0 + X^{-\alpha})\langle\bar\phi^*\rangle \Big)^2$$

$$= \frac{g^2}{2}\Big((\langle\phi^*\rangle X^0 \phi + \phi^* X^0 \langle\phi\rangle - \langle\bar\phi\rangle X^0 \bar\phi^* - \bar\phi X^0 \langle\bar\phi^*\rangle)$$
$$+ 2(\langle\phi^*\rangle X^{+\alpha}\phi - \langle\bar\phi\rangle X^{+\alpha}\bar\phi^*)$$
$$(\phi^* X^{-\alpha}\langle\phi\rangle - \bar\phi X^{-\alpha}\langle\bar\phi^*\rangle) \Big)$$

$$= \frac{g^2 v^2}{2}\Big(\frac{(N-1)}{2(N^2-N)}\left(h + h^* - (\bar{h}^* + \bar{h})\right)^2$$
$$+ (H^\alpha - \bar{H}^{*\alpha})(H^{*\alpha} - \bar{H}^\alpha) \Big)$$

$$H^{+\alpha} = \frac{1}{\sqrt{2}}(H^\alpha - \bar{H}^{*\alpha}) \qquad \Pi^{+\alpha} = \frac{1}{\sqrt{2}}(H^\alpha + \bar{H}^{*\alpha})$$

$$H^{-\alpha} = \frac{1}{\sqrt{2}}(H^{*\alpha} - \bar{H}^\alpha) \qquad \Pi^{-\alpha} = \frac{1}{\sqrt{2}}(H^\alpha + \bar{H}^\alpha)$$

$$h^0 = \mathrm{Re}(h - \bar{h}) \qquad \Pi^0 = \mathrm{Im}(h - \bar{h})$$
$$\Sigma = \frac{1}{\sqrt{2}}(h + \bar{h}) \text{ complex}$$

$$V_{mass} = g^2 v^2 \left(\frac{N-1}{N} h^{0\,2} + H^{+\alpha} H^{-\alpha} \right)$$

$\sigma, \bar{\sigma}, \pi^+, \pi^-, \pi^0, \Omega$ $\quad m = 0$

$h^0 \quad m = gv\sqrt{\frac{2(N-1)}{N}}$

$H^+, H^- \quad m = gv$

remove $\pi^0, \pi^{+\alpha}, \pi^{-\alpha}$
by going to unitary gauge

$$G^B A^B_\mu = X^0 W^0_\mu + X^{+\alpha} W^{+\alpha}_\mu + X^{-\alpha} W^{-\alpha}_\mu + T^a A^a_\mu$$

$$\mathcal{L}_{A^2\phi^2} = g^2 A^A_\mu A^B_\nu \langle \phi^* \rangle G^A G^B \langle \phi \rangle g^{\mu\nu}$$

$$= g^2 \langle \phi^* \rangle \left(X^0 W^0_\mu X^0 W^0_\nu + X^{+\alpha} W^{+\alpha}_\mu X^{-\alpha} W^{-\alpha}_\nu \right) \langle \phi \rangle g^{\mu\nu}$$

$$= g^2 v^2 \left(\frac{(N-1)^2}{2(N^2-N)} W^0_\mu W^{0\mu} + \frac{1}{2} W^{+\alpha} W^{-\alpha} \right)$$

$$m_{W^0} = gv\sqrt{\frac{2(N-1)}{N}} \qquad m_{W^\pm} = gv$$

$V=0$	$SU(N)$	$SU(F)$	$SU(F)$	# B
A^a, λ^a	Ad	1	1	$2(N^2-1)$
ϕ, Q	$\square$	$\square$	1	$2FN$
$\bar{\phi}, \bar{Q}$	$\bar{\square}$	1	$\bar{\square}$	$2FN$

$V\neq 0$	$SU(N-1)$	$SU(F-1)$	$SU(F-1)$	
$m=gv\sqrt{\frac{2(N-1)}{N}}$ $\begin{pmatrix} W^0 & h^0 \\ \lambda^0, & \frac{\omega^0-\bar{\omega}^0}{\sqrt{2}} \end{pmatrix}$	1	1	1	4
$m=gv$ $\begin{pmatrix} W^+ & H^+ \\ \lambda^+ & \omega \end{pmatrix}$	$\square$	1	1	$(3+1)(N-1)$
$\begin{pmatrix} W^- & H^- \\ \lambda^- & \bar{\omega} \end{pmatrix}$	$\bar{\square}$	1	1	$(3+1)(N-1)$

$m=0$				
A, λ	Ad	1	1	$2((N-1)^2-1)$
$\phi' \; Q'$	$\square$	$\square$	1	$2(F-1)(N-1)$
$\bar{\phi}' \; \bar{Q}'$	$\bar{\square}$	1	$\bar{\square}$	$2(F-1)(N-1)$
$\sigma \; \psi$	1	$\square$	1	$2(F-1)$
$\bar{\sigma} \; \bar{\psi}$	1	1	$\square$	$2(F-1)$
$\frac{h+\bar{h}}{\sqrt{2}}, \frac{\omega^0+\bar{\omega}^0}{\sqrt{2}}$	1	1	1	2

$$4 + 4(N-1) + 4(N-1) + 2(N^2-2N+1-1) + 4(F-1)(N-1) + 4(F-1) + 2$$

$$= 8N - 4 + 2N^2 - 4N + 4FN - 4F - 4N + 4 + 4F - 4 + 2$$

$$= 2(N^2-1) + 4FN$$